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Hysteresis in neural networks

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The zero-temperature dynamics of neural networks is used to study history-dependent effects in random magnets. Random-field as well as random-bond Ising models are shown to exhibit hysteresis. The hysteresis loops of the random-bond model have a staircase structure while those of the random-field model are relatively smooth. We discuss the relevance of neural networks in the context of random-bond models, and the bearing the neural network dynamics has on the spectrum of relaxation times in a hysteretic system.

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I. INTRODUCTION

History-dependent effects occur in a wide variety of materials. A familiar example is hysteresis in magnets. As the applied field is swept from a large negative value (saturating the magnetization to its full negative value) to a large positive value and back, the magnetization curve makes a loop. Thus the induced magnetization does not depend on the applied field alone, but also on the previous condition of the magnet. If the magnet was previously situated in a larger magnetic field it yields a larger magnetization. Similar but more complex history-dependent effects are seen in spin glasses [1] as well. The theoretical understanding of history-dependent effects is still in its infancy. These effects are sometimes thought to be nonequilibrium phenomena, and this is seen as the primary difficulty in understanding them on the basis of equilibrium statistical mechanics. In the present work we adopt the viewpoint that systems showing history-dependent effects have several equilibrium states. We view each equilibrium state as an attractor under the relaxational dynamics of the system. The configurational states of the system form a number of nonoverlapping domains. Each domain includes an equilibrium state, and other states which relax into the equilibrium state under the system's dynamics. Initial configurations lying in separate domains produce distinct equilibrium states. This is the origin of the history-dependent effects in our scheme. The initial states can be varied by preparing the system differently or by tuning a relevant parameter of the system such as the applied field.

In order to proceed further we need a model of the system as well as its relaxational dynamics. For this purpose we adopt the Hopfield model [2] of a neural network and its zero-temperature parallel dynamics. The motivation for this choice is as follows. History-dependent effects are observed at relatively low temperatures. This means that the thermal energy of a hysteretic system is small in comparison with other relevant energy scales. The relevant energy scales are the barriers between different equilibrium states and these are apparently controlled by the amount of disorder in the system. In hysteretic systems the relaxation appears to be dominated by disorder effects rather than thermal effects. The zero-temperature parallel dynamics suffices as a minimal model for studying the key features of hysteresis. The restriction to parallel dynamics is not serious although it greatly simplifies analytical as well as numerical calculations. The main difference between parallel and sequential dynamics is that the sequential dynamics takes about N (number of spins) steps more for each step of the parallel dynamics, and always ends in a fixed point configuration. This is because the sequential dynamics flips a spin only if it lowers the energy of the system. However, it is possible to modify the parallel dynamics such that it too yields only fixed points.

The motivation for modeling the system by a neural network is as follows. Experience with spin glasses has shown that random systems having several equilibrium states have conflicting interactions, and the equilibrium states have no apparent long-range order. It is difficult to have direct knowledge of interactions in a random system. It is more convenient to mathematically construct the interactions by the superposition of a number of random configurations of the system as is done in the neural network models of associative memory. The neural network dynamics yields equilibrium configurations which are quite similar to the ones used for the construction of the interactions. We consider neural network models based on Ising spins. The pair interactions between spins are taken to be random in sign and magnitude, and infinite range in the sense that every spin interacts with every other spin. The equilibrium states have no net magnetization in the absence of an applied field, i.e., as many spins are up as down. Some reflection shows that this is not an inappropriate model for a ferromagnet having several magnetized domains, but with net magnetization zero. We can think of all the up spins as belonging to one class, and the down spins as belonging to another class. The effective interactions within each class are ferromagnetic and across the two classes antiferromagnetic. The number of ferromagnetic and antiferromagnetic interactions are approximately equal. This is just another way of looking at the energetics of domain formation. We recall that the physical reason for the formation of domains is the competition between the exchange forces which prefer to have all spins parallel and the long-range dipolar forces which prefer distant regions to be magne-

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tized opposite to each other. The number of domains in equilibrium are determined by the balance between the magnetic energy contained within domains and the energy across domain walls.

II. RELAXATION DYNAMICS

The system is characterized by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij}s_i s_j - \sum_i f_i s_i(t) - H\sum_i s_i(t) \; . \quad (1)$$

Here $\{s_i = \pm 1\}$ are N Ising spins with pair interactions J_{ij}, f_i is a random field (included in order to compare our results with the work on the random-field Ising model), and H is the applied field. The pair interactions are generated by the superposition of p uncorrelated patterns,

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{P} \xi_i^\mu \xi_j^\mu (1-\delta_{ij}) \; , \quad (2)$$

where ξ_i^μ are quenched random Ising variables with equal probability of being ± 1. The spins are updated in parallel by the rule

$$s_i(t+1) = \mathrm{sgn} h_i(t), \qquad h_i(t) = -\sum_j J_{ij}s_j(t) - f_i - H \; . \quad (3)$$

Here h_i is the local field at the ith site. We find it useful to define a quantity,

$$G(t) = -\sum_i |h_i(t)| + \sum_i f_i s_i(t) + H\sum_i s_i(t) \; . \quad (4)$$

The quantity $G(t)$ is somewhat analogous to the zero-temperature Gibbs free energy in the sense that the first term is similar to the internal energy of the system and the last two terms constitute the potential energy of the magnet in the applied field. The quantity $G(t)$ has the remarkable property that it is a decreasing function of time under the parallel dynamics of the system. In this sense it is indeed like a free energy, and the spin update rule mimics the true relaxational dynamics of a physical system because it takes the system towards lower free energy. It can be shown that

$$G(t+1) - G(t) = \sum_i |h_i(t+1)|\{s_i(t)s_i(t+2) - 1\} \; . \quad (5)$$

The derivation of Eq. (5) is independent of the specific form of the pair interactions and assumes only that they are symmetric, i.e., $J_{ij} = J_{ji}$. It proves that $G(t)$ is a decreasing function of time because the first factor on the right-hand side is positive and the second factor in the curly braces can only take the values 0 or -1. Thus at each step the dynamics takes the system to a lower value of $G(t)$.

An interesting result which can be deduced from Eq. (5) is that the end result of the dynamics is a fixed point or a limit cycle of period two in the configuration space. For a finite number of spins, $G(t)$ has a lower bound and therefore the dynamics cannot lower G indefinitely. It must end at a locally minimum value of G, say G^*. Equation (5) requires that when $G(t+1) = G(t) = G^*$ we must have $s_i(t)s_i(t+2) = 1$, or equivalently $s_i(t+2) = s_i(t)$ at each site. This means that the fixed point G^* corresponds to a limit cycle of period two in the configuration space. Note that a limit cycle of period two does not necessarily mean that $s_i(t+1) = -s_i(t)$ for each i. A fraction of the spins may remain frozen, i.e., $s_i(t) = s_i(t+1) = s_i(t+2)$ for some sites i. A fixed point is a special case of a limit cycle of period two when $s_i(t) = s_i(t+1) = s_i(t+2)$ for each site i. The dynamical equation (3) does not give a clear prescription of how to update a spin in zero local field. In this case we may supplement this equation by requiring either (i) $s_i(t+1) = s_i(t)$, or (ii) $s_i(t+1) = -s_i(t)$, or (iii) $s_i(t+1) = 0$. For any of these choices the final state of the dynamics is a fixed point or a limit cycle of period two. We also note that in the case of totally antisymmetric pair interactions, i.e., $J_{ij} = -J_{ji}$, the inequality $G(t+1) \leq G(t)$ is still valid but we obtain a limit cycle of period four. It follows from the dynamical equations that the mirror image of every fixed point (limit cycle) is also a fixed point (limit cycle). The two component states of a limit cycle of period two have the same G^*. We exploited this property to modify the parallel dynamics with an additional rule that the configurations would be updated only if $G(t+1) < G(t)$, but not if $G(t+1) = G(t)$. This modified parallel dynamics gives only fixed points as does the sequential dynamics. It is of course more efficient than the sequential dynamics because the sequential dynamics requires about N steps for each step of the parallel dynamics. Therefore we have adopted the modified parallel dynamics in our numerical work. However, all choices of the dynamical update rule yield qualitatively similar results.

III. HYSTERESIS

Having described the general features of the model dynamics, we now apply it to study hysteresis. As indicated earlier, hysteresis arises in systems having several equilibrium states. We are not aware of any general criteria which govern the number of equilibrium states of a system, but we note that strongly disordered systems such as spin glasses have a large number of nearly degenerate equilibrium states separated by high energy barriers. We limit ourselves to the study of disorder-driven hysteresis only based on the random-field and random-bond Ising models. First we consider the case of site disorder. Let $p = 1$, i.e., let only one configuration be stored. Let this configuration have all spins up. In view of the mirror image symmetry the complementary configuration having all spins down is automatically stored in the interaction matrix J_{ij}. This interaction favors a ferromagnetic state with all spins up or all down. An on-site random field is incorporated through the f_i term in Eq. (1). We will not discuss this case in detail because it is very similar to the work of Sethna *et al.* [3]. However, a remark is in order. Reference [3] demonstrates hysteresis in the nearest neighbor random-field Ising model on a simple cubic lattice by numerical simulation of the model. A mean field theory based on the infinite-range interaction does not show any hysteresis. This may create an erroneous impression that hysteresis is a property of short-range interactions only and lost in models where a spin interacts with every other spin. Our numerical simulations of the neural network model with $p = 1$ and random fields produce hysteresis loops almost identical to those reported in Ref. [3]. Thus a random-field Ising model in which a spin interacts with all other spins shows hysteresis. The problem is not with the range of interactions but with the method of the signal-to-noise ratio on which the mean field theory is based [4]. This method fails when the noise is greater than the signal.

Next we consider the random-bond Ising model. Random bonds are generated in accordance with Eq. (2) by superposing p configurations. Again, one configuration is chosen to have all spins up. This amounts to having an interaction $1/N$ between each pair of spins. It favors a ferromagnetic ground state with all spins up or all spins down. There is no randomness in the model so far and no hysteresis. Randomness is provided by $p - 1$ other configurations which are chosen randomly. Increasing p increases randomness in the model and gives rise to more pronounced hysteretic effects. The solid curve in Fig. 1 shows a typical hysteresis loop for $p = 2N$ ($N = 200$). This corresponds to a relatively large disorder in the system. Other values of p yield similar results. The hysteresis curve is obtained as follows. Starting from a random initial configuration, the system is allowed to evolve under the parallel dynamics in a sufficiently large negative field. In one or two iterations the system settles into a fixed point configuration having all spins down as expected. Next the field is increased in a suitable step and the previous fixed point state is used as the initial state. After a few iterations a new fixed point configuration is obtained and the magnetization $M = \frac{1}{N}\sum_i s_i$ of this configuration is plotted against the applied field. In this way the field is swept up to a sufficiently high value (yielding a fixed point state with all spins up), and back.

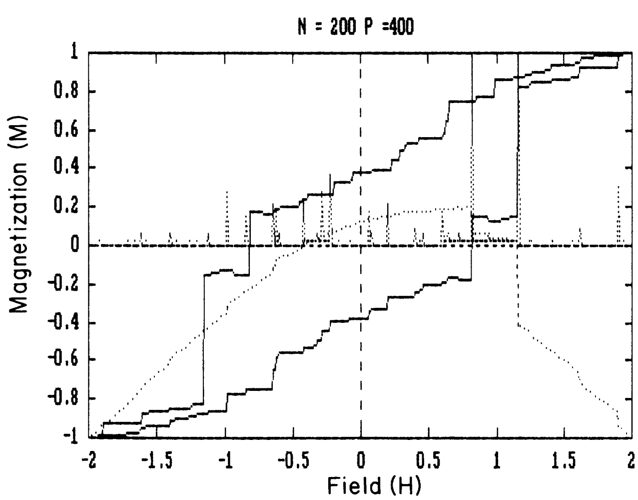

FIG. 1. Hysteresis in a neural network of 200 Ising spins with 400 stored states. The bold curve shows the magnetization; lower (upper) portion corresponds to increasing (decreasing) field. The broken line shows the number of iterations required to reach the final state in increasing field. The dotted line shows the variation of the fixed point energy G^* in the increasing field.

A prominent feature of the hysteresis loop is its staircaselike structure. The vertical steps are not exactly vertical but only approximately so. The graph shows magnetization at successive intervals of the applied field. In any one portion of the hysteresis loop pertaining to either increasing or decreasing field there are nowhere two values of the magnetization for the same field, i.e., no exactly vertical steps. Nevertheless the variation in magnetization occurs in the form of very steep climbs over small increments of the applied field. The nearly vertical steps are separated by longer spans of applied field with the magnetization remaining nearly constant. A closer examination of the hysteresis loop reveals an exact symmetry between the two halves of the loop corresponding to increasing and decreasing field, respectively. This symmetry follows from the invariance of the dynamical equations when all the spins are flipped and the sign of the applied field is also reversed. Other attributes of the graph such as the number of iterations taken to arrive at the fixed point configuration and the free energy G^* of this configuration possess the same symmetry. Therefore we have shown these quantities in Fig. 1 for increasing field only. The broken curve in Fig. 1 shows the number of iterations taken by the dynamics to reach the fixed point configuration. This number varies from 1 to 35 but we have scaled it down to the range 0 to 1 so that it can share a common y axis scale along with magnetization. The number of steps taken to reach the final state is normally one or two except at the near-vertical steps of the hysteresis curve where it takes longer to reach the final state. Larger jumps in magnetization generally require a larger number of iterations. The distribution of the number of iterations has the same significance as the spectrum of relaxation times in a physical system. Our model predicts that the experimental relaxation times will be longest in the region where the magnetization switches over from negative values to positive values in increasing field. As far as we are aware no systematic experimental study has been reported concerning the relaxation times of a hysteretic system in different applied fields. It is possible that even the longest relaxation times may be too short for ordinary experiments to resolve the spectrum of relaxation times even if one existed. The dotted curve in Fig. 1 shows the equilibrium free energy G^* at various values of the field. G^* varies approximately from -400 to 25, but again the range has been scaled down to -1 to 1 so that it can share the y-axis scale for magnetization.

IV. CONCLUDING REMARKS

We conclude that randomness may be the key to understanding history-dependent effects in materials. The study of random-field as well as random-bond Ising models reveals that strongly disordered systems are inherently hysteretic. The random-field model gives a relatively smooth hysteresis curve very similar to what is observed in a typical laboratory experiment. On the other hand, the random-bond model gives a hysteresis curve with an apparently staircase structure. The randomness in the random-field model is independent of the spin configuration of the system and therefore it does not change during the relaxation of the system from an initial state to its final equilibrium state. This feature of the model is somewhat unphysical. On the other hand, in the random-bond model the random field acting on a spin depends on the state of the other spins as well and changes during the dynamical evolution of the system. The random-bond model appears to be more physical from this point of view but surprisingly it is the random-field model which gives more familiar hysteresis curves. This result suggests that in a typical magnet showing hysteresis the random field acting on an atomic magnetic moment may arise from the disorder in its immediate environment (containing perhaps nonmagnetic moment bearing atoms) and the effect of other magnetic moments through exchange forces may be of little importance. The random-field model shows return-point memory as well. This has been explained theoretically on the basis of Middleton's no passing theorem [5]. Middleton's theorem guarantees return-point memory if the dynamics is adiabatic. However, some reflection would convince the reader that Middleton's theorem fails in the case of a random-bond model if the sign of the bonds is random. One has to look for an alternate theoretical basis for analyzing the random-bond model. We have proposed a free energy like quantity G which decreases under the parallel dynamics for the random-field as well as the random-bond model. This helps us to understand several aspects of the numerical simulations, e.g., the symmetry of the hysteresis loop, and also why the dynamical trajectories end in a fixed point or a limit cycle of period two, but are never chaotic. The limit cycles can be eliminated by a slight modification of the parallel dynamics leaving only fixed point configurations as the equilibrium states. It not only brings the simulations closer to the hysteretic behavior of physical systems but also illustrates the primary importance of the relaxation dynamics in determining the energy barriers between different equilibrium states of a disordered system. In short, the modified parallel dynamics of Ising neural networks provides a fairly useful laboratory for studying history-dependent effects in disordered systems.

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